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ON THE OPTIMAL INVENTORY EQUATION

Richard Bellman
Irving Glicksberg
Oliver Gross

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary

~~In this paper we present~~ a summary IS PRESENTED of some results recently obtained in connection with the problem of optimal inventory. Solutions are given for the case of proportional costs and arbitrary demand, and for the case where the costs are linear with constant terms representing administrative costs.( )

ON THE OPTIMAL INVENTORY EQUATION

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§1. Introduction

The purpose of this note is to present some results concerning the functional equations

$$f(x) = \underset{y \geq x}{\mathrm{Min}} \left[g(y-x) + a \left[\int_y^{\infty} p(s-y)\,dG(y) + f(0) \int_y^{\infty} dG(s) + \int_0^y f(y-s)\,dG(s) \right] \right] \tag{1.1}$$

and

$$\begin{aligned} f(x_1, x_2) = \underset{\substack{y_1 \geq x_1 \\ y_2 \geq x_2}}{\mathrm{Min}} \Bigg[& g_1(y_1-x_1) + g_2(y_2-x_2) + a \Bigg[\int_{y_1}^{\infty} \int_{y_2}^{\infty} \big[p_1(s_1-y_1) \\ & \qquad + p_2(s_2-y_2) + f(0,0) \big] dG(s_1, s_2) \\ & + \int_{y_1}^{\infty} \int_0^{y_2} \big[p_1(s_1-y_1) + f(0, s_2-y_2) \big] dG(s_1, s_2) + \int_0^{y_1} \int_{y_2}^{\infty} \big[f(y_1-s_1, 0) \\ & \qquad + p_2(s_2-y_2) \big] dG(s_1, s_2) \\ & + \int_0^{y_1} \int_0^{y_2} f(y_1-s_1, y_2-s_2)\,dG(s_1, s_2) \Bigg] \Bigg] \end{aligned} \tag{1.2}$$

which are respectively one- and two-dimensional versions of equations arising from the "optimal inventory" problem, an interesting and important economic and industrial problem. We will state one version of this below.

The mathematical formulation of the problem follows that of Arrow, Harris and Marschak,[1]. A more detailed study, containing existence and uniqueness theorems and a discussion of related statistical questions, is contained in Dvoretzky, Kiefer and Wolfowitz,[5], cf. also Bellman,[4].

Up to the present time, however, no general analytic solution of (1) has been given, and it would seem that a solution for arbitrary functions must possess a quite complicated and intricate structure. Here we shall present a solution for the case where functions, $g(y)$ and $p(y)$ are linear functions of y, but where the distribution function dG is arbotrary.

The method we use is equally applicable to a number of other cases as well, as we shall indicate briefly below. A detailed account of these results is in the course of preparation.

§2. The "Optimal Inventory" Problem.

One version of the optimal inventory problem is the following. At any time we have N resources in quantities $x_1, x_2, \ldots, x_N$. We may, if we wish, order additional quantities of these resources so that we have on hand quantities $y_1, y_2, \ldots, y_N$, at a cost of $\sum_{i=1}^{N} g_i(y_i - x_i)$. Having had this opportunity to increase our supplies, we face an unknown demand $(s_1, s_2, \ldots, s_N)$, where s_i is the demand for the i-th resource, with a known distribution function $dG(s_1, \ldots, s_N)$ This demand is met as far as possible. If, however, $s_i > y_i$, we must order a quantity $s_i - y_i$ at higher prices, to meet this demand. Assuming a constant discount factor a in the cost of ordering at each subsequent stage, how do we order at each stage so as to minimize the total expected cost over time?

In the one-dimensional case where x is the quantity of the one resource on hand at any particular time, and $f(x)$ is total expected cost using an optimal ordering policy, it is readily verified that $f(x)$ satisfies (1.1). Similarly, $f(x_1,x_2)$ the corresponding function in the two-resource case, satisfies (1.2).

As problems in the theory of dynamic programming, see, [2], and [3], the important aspect of the solution is not so much the cost function $f(x)$ as the structure of the policy function $y(x)$, the amount ordered at each stage. Using techniques employed in the treatment of functional equations of similar type, see [2], [4], we obtain the solutions to the above equations under the assumptions mentioned above and discuss below solutions obtained under some modified assumptions.

§ 3. The Solution of the N-dimensional case.

In presenting the solution of the N-dimensional case, we shall state the result in two-dimensional terms for ease of notation, and shall make some inessential simplifications which permit a simple statement of the results.

Theorem. Let us assume that

$$
\begin{aligned}
&(a)\quad g_i(y) = k_i y,\ p_i(y) = p_i y, \text{ where } k_i \text{ and } p_i \text{ are positive constants.}\\
&(b)\quad dG(s_1,s_2) = \phi(s_1,s_2)ds_1 ds_2,\ \phi > 0.\\
&(c)\quad \int_0^\infty\!\!\int s_i\phi(s_1,s_2)ds_1 ds_2 < \infty \quad, i = 1,2.\\
&(d)\quad 0 < a < 1.
\end{aligned}
\tag{3.1}
$$

Then the solution to (1.2) is as follows. Let x_i, $i = 1,2$, be defined as the unique positive solutions of

$$k_1 - p_1 a \int_{\bar{x}_1}^{\infty} \left(\int_0^{\infty} \phi(s_1, s_2)\, ds_2 \right) ds_1 - k_1 a \int_0^{\bar{x}_1} \left(\int_0^{\infty} \phi(s_1, s_2)\, ds_2 \right) ds_1 = 0,$$

$$k_2 - p_2 a \int_{\bar{x}_2}^{\infty} \left(\int_0^{\infty} \phi(s_1, s_2)\, ds_1 \right) ds_2 - k_2 a \int_0^{\bar{x}_2} \left(\int_0^{\infty} \phi(s_1, s_2)\, ds_1 \right) ds_2 = 0. \qquad (3.2)$$

provided that they exist. Then

(a) For $0 \leq x_i \leq \bar{x}_i$, we choose $y_i = \bar{x}_i$.
(b) For $x_i \geq \bar{x}_i$, we choose $y_i = x_i$.

If x_i does not exist, the above prescription is valid with $x_i = 0$.

Observe that the form of the solution is independent of the dimension, and has, in addition, the important property of sub-optimality.

§ 4. Idea of the Proof for the One-Dimensional Case.

Let us indicate briefly the essence of the proof, using the one-dimensional case as our model. The key to the mathematical analysis is the observation that in the equation

$$u(x) = \min_{y>x} v(x,y), \qquad (4.1)$$

whenever the minimum occurs at a finite value $y>x$, we have, for this value of y the simultaneous equations

$$v_y(x,y) = 0, \quad u'(x) = v_x(x,y) \qquad (4.2)$$

Utlizing the observation, we see that, under the linearity assumptions of the above theorem, whenever $y>x$ we have simultaneously

$$(a)\ k_1 - ap_1 \int_{y}^{\infty} \phi(s)\,ds + \int_{0}^{y} f'(y-s)\phi(s)\,ds = 0 \qquad (4.3)$$

$$(b)\ f'(x) = -k_1.$$

Upon these two equations, the remainder of the proof hinges.

§5. Further Results.

The same techniques, reinforced by strenuous application of the method of successive approximations, are applicable to more general processes, involving nonlinear cost and penalty functions, time-lags, and so forth.

A case of particular importance is that where the cost function has the form ky + R, where y is the amount ordered and R is a fixed "red tape" cost. If the penalty cost retains its proportional form, then regardless of the distribution of demand, the optimal policy is an "sS-policy" of the type discussed by Arrow, Harris and Marschak, [1],. This is also true if we allow the penalty cost to contain a similar red-tape term, for a large class of distribution functions of practical importance.

These results, together with proofs and extensions of the above results will appear in a detailed exposition to appear elsewhere.

1. Arrow, K. J., Harris, T. E., and Marschak, J., "Optimal Inventory Policy", Econometrica, July, 1952.
2. Bellman, R., "An Introduction to the Theory of Dynamic Programming", Rand Report 245, 1953.
3. Bellman, R., "The Theory of Dynamic Programming", Bull. Amer. Math. Soc., (to appear).
4. Bellman, R., "The Optimal Inventory Equation", Rand Paper 430, 1954
5. Dvoretzky, A. J., Kiefer J. and Wolfowitz, J., "The Inventory Problem, I, II, Econometrica, 1952, pp 187-222.